Lecture 17

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March 12, 2003

1 Kernel: its dimension and basis

Last lecture we saw that the kernel of a linear function is a vector space. Each vector space has a dimension and basis — this lecture we'll try to determine them for the kernel.

Let $f : V \to U$ be a linear function, V and U are vector spaces, $\dim V = n$, and $\dim U = m$. We saw that for any linear function we can determine its matrix. So, for f there exists $m \times n$-matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

such that for any vector $x \in V$

$$f(x) = Ax,$$

i.e. if x is a vector such that $x = (x_1, x_2, \dots, x_n)$, then

$$f(x) = Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

By definition, vector x belongs to the kernel of f if and only if $f(x) = \mathbf{0}$. But since $f(x) = Ax$, then this condition can be written as

$$f(x) = 0 \iff Ax = 0 \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So, the last matrix equality can be written as a linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

This is a homogeneous system, and we got that its solution space is the kernel of f. We already know how to get a dimension and basis of the solution set for any homogeneous system. The dimension of it will be equal to the number of free variables in REF, and to find a basis we have to solve it, and get particular solutions by assigning 1 to the first free variable and 0's to the other, then 1 to the second free variable and 0's to the other, etc. These solutions will constitute a basis of the solution space.

Example 1.1. *Let $f\mathbb{R}^4 \to \mathbb{R}^3$ be a linear function such that*

$$f(x, y, z, u) = (x + 2y - z + u,\ 2x + y + z + u,\ 4x + 5y - z + 3u).$$

We can write the system for kernel:

$$\begin{cases} x + 2y - z + u = 0 \\ 2x + y + z + u = 0 \\ 4x + 5y - z + 3u = 0 \end{cases}$$

Let's transpose it to REF:

$$\begin{cases} x + 2y - z + u = 0 \\ 2x + y + z + u = 0 \\ 4x + 5y - z + 3u = 0 \end{cases} \rightsquigarrow \begin{cases} x + 2y - z + u = 0 \\ - 3y + 3z - u = 0 \\ - 3y + 3z - u = 0 \end{cases} \rightsquigarrow \begin{cases} x + 2y - z + u = 0 \\ - 3y + 3z - u = 0 \end{cases}$$

Here we have 2 free variables, so the dimension of the $\operatorname{Ker} f$ *is equal to 2. To find the basis of it we need to find 2 particular solutions of it. Free variables are z and u. So, first, assign $z = 1$ and $u = 0$. We get: $y = 1$ and $x = -2y + z - u = -2 + 1 = -1$. So, the first solution is $(x, y, z, u) = (-1,\ 1,\ 1,\ 0)$. Second, assign $z = 0$ and $u = 1$. We get: $y = -1/3$, and $x = -2y + z - u = 2/3 - 1 = -1/3$. So, the second solution is $(x, y, z, u) = (-1/3,\ -1/3,\ 0,\ 1)$. So, the basis of the kernel consists of the following 2 vectors:*

$$e_1 = (-1,\ 1,\ 1,\ 0)$$
$$e_2 = (-1/3,\ -1/3,\ 0,\ 1).$$

Now we'll develop a theory about the dimension of the kernel. Let consider the $m \times n$ matrix in the REF. We see that the variable can be either leading or free. So, since the total number of variables is equal to n, we have the following:

$$\#\text{ free variables} + \#\text{ leading variables} = n.$$

Number of leading variables is equal to the rank of a matrix (since each leading variable corresponds to one nonzero row in REF of a matrix), and the number of free variables as we saw is equal to the dimension of the kernel of f. So, we get the following lemma:

Lemma 1.2. *If $f : V \to U$ is a linear function with matrix A, and $\dim V = n$, then*

$$\dim \operatorname{Ker} f + \operatorname{rk} A = n.$$

2 Image: its dimension and basis

Last lecture we saw that if for a linear function $f : V \to U$ we know $f(e_1), f(e_2), \ldots, f(e_n)$, $(e_1, e_2, \ldots, e_n$ is a basis of $V)$ then we can determine $f(v)$ for any vector $v \in V$. I.e., if

$$v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

then

$$f(v) = a_1 f(e_1) + a_2 f(e_2) + \cdots + a_n f(e_n).$$

So, vectors $f(e_1), f(e_2), \ldots, f(e_n)$ span the image of f. Thus, the problem of finding the dimension and the basis of the image reduces to the problem of finding the dimension and the basis of the span of $f(e_1), f(e_2), \ldots, f(e_n)$. We already know how to solve this problem: we write vectors $f(e_1), f(e_2), \ldots, f(e_n)$ as rows of a matrix, reduce it to REF, and the number of nonzero rows is equal to the dimension, and nonzero rows constitutes a basis of it.

Example 2.1. *We'll find the dimension and basis of the image of the same linear function as we considered in the previous example.*

$$f(x, y, z, u) = (x + 2y - z + u,\ 2x + y + z + u,\ 4x + 5y - z + 3u).$$

We have:

- $e_1 = (1, 0, 0, 0) \Rightarrow f(e_1) = (1, 2, 4)$
- $e_2 = (0, 1, 0, 0) \Rightarrow f(e_2) = (2, 1, 5)$
- $e_3 = (0, 0, 1, 0) \Rightarrow f(e_3) = (-1, 1, -1)$
- $e_4 = (0, 0, 0, 1) \Rightarrow f(e_4) = (1, 1, 3)$

So, let's write them as rows of a matrix and transpose it to REF:

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we see that the number of nonzero rows is 2 — it's a dimension of the image, and its basis consists of the following 2 vectors:

$$e_1 = (1, 2, 4)$$
$$e_2 = (0, -3, -3).$$

3 Extension to basis

Now we will jump back to vector spaces, and consider the following problem. Let's suppose that we have a vector space V, such that $\dim V = n$, and we're given m linearly independent vectors, such that $m < n$. As we know, the basis of V consists of n vectors. So, our problem is to find $(n - m)$ vectors, such that these found vectors and given vectors together form a basis of V.

To solve this problem we will use the following algorithm. Let's take any basis of V (for example, a standard basis). Then we will try to add vectors from it one by one to the given set of vectors and see whether they together form an independent set. If yes, we keep it and include it to the new basis, if no, we simply drop it. By the end of this procedure we will have n vectors which form a basis.

Example 3.1. *Consider the space $\mathbb{R}^3$. Let we have only one vector $u_1 = (4, 2, 0)$. We want to find a basis containing this vector u_1.*

Consider the standard basis:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We'll check vectors from it one by one.

Step 1. *Vector e_1. Let's check whether u_1 and e_1 are linearly independent. Let's make a linear combination which is equal to 0.*

$$x \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can write it as a system of linear equations:

$$\begin{cases} 4x + y = 0 \\ 2x = 0 \end{cases}$$

This system has the unique solution $x = 0,\ y = 0$. So, these vectors are linearly independent and we can add vector e_1.

Step 2. *Vector e_2. Let's check whether vectors u_1, e_1 and e_2 are linearly independent. Let's make a linear combination which is equal to 0.*

$$x \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can write it as a system of linear equations:

$$\begin{cases} 4x + y = 0 \\ 2x + z = 0 \end{cases}$$

This system is homogeneous system, in which the number of equations if less than the number of variables. So, this system has nonzero solution, e.g. $x = 1$, $y = -4$, and $z = -2$ (we should not really find it — it's enough to know that it exists!!!). So, we drop the vector e_2.

Step 3. *Vector e_3. It's the last vector, and we should not even check if u_1, e_1 and e_3 are independent — we can just simply include it into the basis. But we'll perform this check as an example of determining linear independence. Let's make a linear combination which is equal to 0.*

$$x \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We can write it as a system of linear equations:

$$\begin{cases} 4x + y = 0 \\ 2x = 0 \\ z = 0 \end{cases}$$

This system has unique zero solution, so these vectors are linearly independent.

Finally, we found a basis of $\mathbb{R}^3$, which contains the vector u_1. It consists of vectors u_1, e_1 and e_3:

$$u_1 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$